Entropy and Area of Black Holes in Loop Quantum Gravity

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Abstract

Simple arguments related to the entropy of black holes strongly constrain the spectrum of the area operator for a Schwarzschild black hole in loop quantum gravity. In particular, this spectrum is fixed completely by the assumption that the black hole entropy is maximum. Within the approach discussed, one arrives in loop quantum gravity at a quantization rule with integer quantum numbers n for the entropy and area of a black hole.

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The quantization of black holes was proposed long ago in the pioneering work [1], and from other points of view in [2, 3]. The idea of [1] was based on the intriguing observation [4] that the horizon area A of a nonextremal black hole behaves in a sense as an adiabatic invariant. This last fact makes natural the assumption that the horizon area should be quantized. Once this hypothesis is accepted, the general structure of the quantization condition for large (generalized) quantum numbers N gets obvious, up to an overall numerical constant α. The quantization rule should be

$$A_N = \alpha\, l_p^2\, N. \tag{1}$$

Indeed, the presence of the Planck length squared

$$l_p^2 = G\hbar/c^3 \tag{2}$$

in formula (1) is only natural. Then, for A to be finite in a classical limit, the power of N in expression (1) should be equal to that of $\hbar$ in l_p^2. This argument, formulated in [5], can be checked, for instance, by inspecting any expectation value nonvanishing in the classical limit in ordinary quantum mechanics.

The subject of the present note is the entropy and spectrum of black holes in loop quantum gravity [6-10]. We confine below to a rather simplified version of this approach where the area spectrum of a spherical surface is

$$A = \alpha\, l_p^2 \sum_{i=1}^{\nu} \sqrt{j_i(j_i+1)}. \tag{3}$$

Here a half-integer or integer "angular momentum" j_i,

$$j_i = 1/2, 1, 3/2, \dots\,, \tag{4}$$

is ascribed to each of ν edges intersecting the surface. This set of edges, labeled by index i, determines the surface geometry. The quantum numbers j_i assigned to these edges are constrained by the condition

$$\sum_{i=1}^{\nu} j_i = n, \tag{5}$$

where n is an integer. To each "angular momentum" $\mathbf{j}_i$ one ascribes $2j_i+1$ possible projections, from $-j_i$ to $+j_i$. We assume below that for a given $\mathbf{j}_i$ all these projections have the same weight. With the vectors $\mathbf{j}_i$ being the only building blocks of the model, it is natural to consider that this $2j_i+1$ degeneracy for each angular momentum $\mathbf{j}_i$ corresponds to the spherical symmetry of the surface.

Some resemblance between expressions (1) and (3) is obvious, though in the last case the large number

$$N = \sum_{i=1}^{\nu} \sqrt{j_i(j_i+1)} \tag{6}$$

is certainly no integer. As to the overall numerical factor α in (3), it cannot be determined without an additional physical input. This ambiguity originates from a free (so-called Immirzi) parameter [11, 12] which corresponds to a family of inequivalent quantum theories, all

of them being viable without such an input. One may hope that the value of this factor in (3) can be determined by studying the entropy of a black hole. This idea (mentioned previously in [13]) is investigated below.

We define the entropy S of a spherical surface as the logarithm of the number of states of this surface with fixed n, ν, and ν_j, where ν_j is the number of edges with given j. Due to the mentioned $2j+1$ degeneracy for each "angular momentum" j, the entropy is

$$S = \ln \left[\prod_j (2j+1)^{\nu_j} \frac{\nu\,!}{\prod_j \nu_j\,!} \right] = \sum_j \nu_j \ln(2j+1) + \ln(\sum_j \nu_j\,!) - \sum_j \ln \nu_j\,!. \tag{7}$$

The obvious constraints are

$$\sum_j \nu_j = \nu; \quad \sum_j j\,\nu_j = n. \tag{8}$$

Let us mention that the entropy arguments exclude for a black hole "empty" edges with $j_i = 0$. Obviously, if "empty" edges were allowed, the entropy would be indefinite even for fixed N and n. In particular, with "empty" edges the Bekenstein-Hawking relation

$$S = \frac{A}{4\,l_p^2} \tag{9}$$

would not hold. Moreover, by adding an arbitrary number ν_0 of "empty" edges in arbitrary order, the entropy could be made arbitrarily large without changing N and n. Indeed, with "empty" edges allowed, the ratio $\nu\,!/\nu_0\,!$ grows indefinitely with ν_0 at fixed values of ν_j with $j \neq 0$.

On the other hand, the same fundamental relation (9) dictates that the number of edges ν should be roughly on the same order of magnitude as the sum n of "angular momenta". Let us mention in this connection the model proposed in [13]. In this model the horizon is characterized by a single edge with $j = n/2$. Then the entropy grows with n logarithmically,

$$S = \ln(2j+1) = \ln(n+1) \to \ln n\,,$$

while the area grows with n linearly,

$$A \sim \sqrt{j(j+1)} \sim \sqrt{n(n+2)} \;\to n\,.$$

Since the requirement (9) for the classical limit $n \to \infty$ is grossly violated in it, the model of [13] has no physical meaning, or at least is incomplete. To save the model, some extra source of degeneracy should be included into it, but one cannot find in [13] any mention of such a degeneracy.

Thus, at least in the approach discussed (as distinct for example from that of [14]), relation (9) is an absolutely nontrivial constraint on a microscopic structure of theory.

It is natural to consider that the entropy of an eternal black hole in equilibrium is maximum. This argument is emphasized in [15], and used therein in a model of the quantum black hole as originating from dust collapse. Just the discussion of the assumption of maximum entropy is the main subject of the present paper. More definite formulation of the problem

considered below is as follows. With the quantum numbers j_i being the only building blocks of the model, we are looking for such their distribution over the edges which results in the maximum entropy for a fixed total amount of the building material $n = \sum j_i$.

It is rather obvious intuitively that the entropy is maximum when all values of j are allowed. To demonstrate that this is correct, we will consider few more and more complex examples step by step, starting with the simplest choice for the quantum numbers j_i, where all of them are put equal to 1/2. Then $\nu_j = \nu\delta_{j,1/2}$, $\nu = 2n$, and

$$S = 2 \ln 2\, n. \tag{10}$$

With all $j_i = 1/2$ and $\nu = 2n$, the area given by formula (3) equals

$$A = \alpha\, l_p^2\, \frac{\sqrt{3}}{2}\, \nu = \alpha\, l_p^2\, \sqrt{3}\, n. \tag{11}$$

Now, under the made assumption we obtain, due to formulae (9) – (11), the following value of the parameter α of the theory:

$$\alpha = \frac{8 \ln 2}{\sqrt{3}}. \tag{12}$$

It should be pointed out that this is the value of the parameter α derived previously in [16] within a Chern-Simons field theory, and that the typical value of j_i obtained therein is also 1/2 (see also [17, 18]).

In fact, in this way one arrives at the quantization rule for the black hole entropy (and area) with integer quantum numbers ν or n (see formula (10)), as proposed in [1]. Moreover, in this picture the statistical weight of the quantum state of a black hole is 2^ν with integer ν, as argued in [2] (in the present case this integer ν should be even).

Let us include now $j = 1$ in line with $j = 1/2$. Then the entropy reaches its maximum value

$$S = 2 \ln 3\, n = 2.197\, n$$

for $\nu_{1/2} = n,\ \nu_1 = n/2$, with the mean value $< j >$ of angular momenta

$$< j >= n/\nu = 2/3 = 0.667.$$

(Here and below we retain in the expressions for entropy only leading terms, linear in the large parameter n.) It is curious to compare these numbers with the analogous ones $S = 2 \ln 2\, n = 1.386\, n$, $< j >= j = 1/2 = 0.5$ for the pure $j = 1/2$ case.

But what happens if quantum numbers larger than 1 are also allowed? When $j = 3/2$ are included, in line with $j = 1/2$ and $j = 1$, the maximum entropy value

$$S = 2.378\, n$$

is attained at $\nu_{1/2} = 0.810\, n,\ \nu_1 = 0.370\, n,\ \nu_{3/2} = 0.150\, n$. Both entropy and average angular momentum

$$< j >= 0.752$$

increase again, but not too much, as compared to the previous ones $S = 2.197\,n$, $< j >= 0.667$.

It is natural now to expect that the absolute maximum of entropy is reached when all values of quantum numbers are allowed. Let us consider this situation starting with the general formula (7). It is convenient to go over in it to new variables y_j:

$$\nu_j = n y_j\,, \tag{13}$$

constrained in virtue of (5) by the obvious relation

$$\sum_j j\, y_j = 1. \tag{14}$$

Then, by means of the Stirling formula for factorials, we transform (7) to the following expression:

$$S = n\left[\sum_j y_j\, \ln(2j+1) + \sum_j y_j \times \ln(\sum_{j'} y_{j'}) - \sum_j y_j \ln y_j\right]. \tag{15}$$

Only the contribution proportional to the large number n is retained here. We have assumed also that the number of essential terms in the sums entering (7) (i. e. the number of the essential classes of the edges with the same j) is much smaller than n. In fact, this number is on the order of $\ln n$, and the leading correction to the approximate formula (15) is on the order of $\ln^2 n$. Again, the situation with the leading correction here is different from that for the case when all $j_i = 1/2$, where the correction is just absent, and from that for the model considered in [17, 18], where it is on the order of $\ln n$.

We are looking for the extremum of expression (15) under the condition (14). The problem reduces to the solution of the system of equations

$$\ln(2j+1) + \ln(\sum_{j'} y_{j'}) - \ln y_j = \mu j, \tag{16}$$

or

$$y_j = (2j+1)\, e^{-\mu j} \sum_{j'} y_{j'}\,. \tag{17}$$

Here μ is the Lagrange multiplier for the constraining relation (14). Summing expressions (17) over j, we arrive at equation

$$\sum_{j=1/2}^{\infty} (2j+1)\, e^{-\mu j} = 1, \quad \text{or} \quad \sum_{p=1}^{\infty} (p+1) z^p = 1, \quad p = 2j\,, \quad z = e^{-\mu/2}\,. \tag{18}$$

Its solution is readily obtained:

$$z = 1 - \frac{1}{\sqrt{2}}\,, \quad \text{or} \quad \mu = -2\ln z = 2.456\,. \tag{19}$$

Let us multiply now equation (16) by y_j and sum over j. Then, with the constraint (14) we arrive at the following result for the absolute maximum of the entropy for a given value of n:

$$S = \mu\, n = 2.456\, n\,. \tag{20}$$

This is the final term of the succession of previous values of entropy S:

$$1.386\,n \quad 2.197\,n, \quad 2.378\,n\,.$$

Assuming that the entropy of an eternal black hole in equilibrium is maximum, we come to the conclusion that it is just (20), which is the true value of the entropy of a black hole

To find the mean angular momentum $< j >$ in the state of maximum entropy, let us rewrite the constraint (14) as

$$y_{1/2} \sum_{p=1}^{\infty} (p+1)p\, z^{p-1} = 1. \tag{21}$$

The sum in the last expression is also easily calculated, and with the value (19) for z we obtain $y_{1/2} = 1/\sqrt{2}$. In its turn, this value of $y_{1/2}$ together with equation (17) gives

$$y_j = \frac{1}{2\sqrt{2}}\,(2j+1) z^{2j-1}, \tag{22}$$

and

$$\sum_{j=1/2}^{\infty} y_j = \frac{\sqrt{2}+1}{2} = 1.207. \tag{23}$$

Now, the mean angular momentum is

$$< j >= \frac{n}{\nu} = \left[\sum_{j=1/2}^{\infty} y_j \right]^{-1} = 2(\sqrt{2}-1) = 0.828, \tag{24}$$

which fits perfectly the succession of previous mean values $< j >$:

$$0.5 \quad 0.667 \quad 0.752\,.$$

Let us come back to the expression (3) for the black hole entropy. The sum (6) is conveniently rewritten as

$$N = \sum_{j=1/2}^{\infty} \sqrt{j(j+1)}\; \nu_j\,.$$

With our formulae (19), (22), one can easily express this sum via n:

$$N = 1.471\, n.$$

Thus, using the Bekenstein-Hawking relation (9), we obtain the following results in the loop quantum gravity for the area A of an eternal spherically symmetric black hole in equilibrium and for the constant α of the area spectrum (3) of a spherical surface:

$$A = 9.824\, l_p^2 n = 6.678\, l_p^2 N; \quad \alpha = 6.678. \tag{25}$$

As to the mass M of a black hole, it is quantized in the units of the Planck mass m_p as follows:

$$M^2 = \frac{0.614}{\pi}\, m_p^2\, n. \tag{26}$$

Let us present also for the sake of comparison the corresponding results of [16]:

$$A_a = 8\ln 2\, l_p^2\, n = \frac{8\ln 2}{\sqrt{3}}\, l_p^2\, N = 3.202 l_p^2\, N; \quad \alpha_a = 3.202; \quad M_a^2 = \frac{\ln 2}{2\pi}\, m_p^2\, n = \frac{0.347}{\pi}\, m_p^2\, n\,.$$

Of course, the solution proposed in [16] looks at least more simple and elegant. On the other hand, the advantage of our solution is that it is based on a simple and natural physical conjecture.

It should be emphasized that in both cases one arrives at the quantization rule for the black hole entropy (and area) with integer quantum numbers n, as proposed in [1].

In conclusion, let us comment briefly upon some previous investigations of the considered problem. In [19] the entropy is defined as logarithm of the number of microstates for which the sum (6) is between N and $N + \Delta N$, $N \gg \Delta N \gg 1$ (here and below the notations of the present article are used). The conclusion made in [19] is that the value of this logarithm is in the interval $(0.96 - 1.38)N$. However, under the only condition $N \gg 1$, without any assumption made about the distribution of the angular momenta j over the edges, how can one arrive at the above numbers $(0.96 - 1.38)N$? The same question (in fact, objection) refers to the results obtained in [20].

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References

[1] J.D. Bekenstein, Lett. Nuovo Cimento 11 (1974) 467.

[2] V.F. Mukhanov, Pis'ma Zh. Eksp. Teor. Fiz. 44 (1986) 50 [JETP Lett. 44 (1986) 63].

[3] Ya.I. Kogan, Pis'ma Zh. Eksp. Teor. Fiz. 44 (1986) 209 [JETP Lett. 44 (1986) 267]; hep-th/9412232.

[4] D. Christodoulu, Phys. Rev. Lett. 25 (1970) 1596;
D. Christodoulu, R. Ruffini, Phys. Rev. D 4 (1971) 3552.

[5] I.B. Khriplovich, Phys. Lett. B 431 (1998) 19; gr-qc/9804004.

[6] C. Rovelli, L. Smolin, Nucl. Phys. B 442 (1995) 593; erratum, ibid. B 456 (1995) 753; gr-qc/9411005.

[7] A. Ashtekar, J. Lewandowski, Class. Quantum Grav. 14 (1997) 55; gr-qc/9602046.

[8] R. Loll, Phys. Rev. Lett. 75 (1995) 3048; gr-qc/9506014;
R. Loll, Nucl. Phys. B 460 (1996) 143; gr-qc/9511030.

[9] R. De Pietri, C. Rovelli, Phys. Rev. D 54 (1996) 2664; gr-qc/9602023.

[10] S. Frittelli, L. Lehner, C. Rovelli, Class. Quantum Grav. 13 (1996) 2921; gr-qc/9608043.

[11] G. Immirzi, Class. Quantum Grav. 14 (1997) L177; gr-qc/9701052.

[12] C. Rovelli, T. Thiemann, Phys. Rev. D 57 (1998) 1009; gr-qc/970559.

[13] M. Bojowald, H.A. Kastrup, Class. Quantum Grav. 17(2000) 3009; hep-th/9907042;
M. Bojowald, H.A. Kastrup, hep-th/9907043.

[14] S.A. Major, K.L. Setter, gr-qc/0101031;
S.A. Major, K.L. Setter, gr-qc/0108034.

[15] C. Vaz, L. Witten, Phys. Rev. D 64 (2001) 084005; gr-qc/0104017.

[16] A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, Phys. Rev. Lett. 80 (1998) 904; gr-qc/9710007;
A. Ashtekar, C. Beetle, S. Fairhurst, Class. Quantum Grav. 16 (1999) L1; gr-qc/9812065;
A. Ashtekar, A. Corichi, K. Krasnov, Adv. Theor. Math. Phys. 3 (2000) 419; gr-qc/9905089;
A. Ashtekar, J. Baez, K. Krasnov, Adv. Theor. Math. Phys. 4 (2001) 1; gr-qc/0005126.

[17] R. Kaul, P. Majumdar, Phys. Rev. Lett. 84 (2000) 5255; gr-qc/0002040.

[18] S. Carlip, Class. Quantum Grav. 17 (2000) 4175; gr-qc/0005017.

[19] C. Rovelli, Phys. Rev. Lett. 77 (1996) 3288; gr-qc/9603063.

[20] K.V. Krasnov, Phys. Rev. D 55 (1997) 3505; gr-qc/960325;
K.V. Krasnov, Gen. Rel. Grav. 30 (1998) 53; gr-qc/9605047;
K.V. Krasnov, Class. Quantum Grav. 16 (1999) 563; gr-qc/9710006.